

NORMAL FORMS FOR SINGULAR HOLOMORPHIC ENGEL SYSTEMS

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ABSTRACT. We prove a normal form theorem for germs of holomorphic singular Engel systems with good conditions on its singular set. As an application, we prove that there exists an integral analytic curve passing through the singular points of the system. Also, we prove that a globally decomposable Engel system on a four dimensional projective space has singular set with atypical codimension.

1. INTRODUCTION

A germ of *holomorphic Pfaff system* of codimension k on $(\mathbb{C}^n, 0)$ is a subsheaf \mathcal{I} of the cotangent sheaf $\Omega_{\mathbb{C}^n}^1$ of $(\mathbb{C}^n, 0)$ and spanned by k germs of holomorphic differential 1-forms $\omega_1, \dots, \omega_k$, which will be denoted by $\mathcal{I} = \langle \omega_1, \dots, \omega_k \rangle$. This system can be represented by the holomorphic k -form $\omega_1 \wedge \dots \wedge \omega_k$. The singular set of \mathcal{I} is the analytic subset given by

$$\text{Sing}(\mathcal{I}) = \{p \in (\mathbb{C}^n, 0); (\omega_1 \wedge \dots \wedge \omega_k)(p) = 0\}.$$

Therefore $\text{Sing}(\mathcal{I})$ is determinantal and has codimension at most $k+1$. We say that the singular set of \mathcal{I} has *expected codimension* if it has a component of codimension $k+1$.

Let $\mathcal{C} = V(\mathfrak{a})$ be a germ of analytic subset on $(\mathbb{C}^n, 0)$ of codimension $\leq k$, with zeros ideal \mathfrak{a} . If $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$, then we denote by $d\mathfrak{a}$ the Pfaff system spanned by df_1, \dots, df_r .

We say that $\mathcal{C} = V(\mathfrak{a})$ is an *integral variety* of $\mathcal{I} = \langle \omega_1, \dots, \omega_k \rangle$ if and only if

$$\omega_i \wedge d\mathfrak{a} \in \mathfrak{a} \otimes \Omega_{\mathbb{C}^n}^{r+1}, \text{ for each } i = 1, \dots, k.$$

A Pfaff system \mathcal{I} is integrable if and only if

$$d\mathcal{I} \equiv 0 \pmod{\mathcal{I}}.$$

By the classical Frobenius's Theorem, for all points $p \in (\mathbb{C}^n, 0) - \text{Sing}(\mathcal{I})$ there exists an integral complex analytic manifold of codimension k passing by p . In [11] B. Malgrange obtained a Frobenius's Theorem for singular integrable systems with singular set of codimension ≥ 3 , showing the existence of integral varieties passing through the singular points of the system.

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For a germ of Pfaff system \mathcal{I} , we can define its *derived flag* $\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \dots$ by the relations $\mathcal{I}^{(0)} = \mathcal{I}$ and

$$\mathcal{I}^{(i+1)} = \{\alpha \in \mathcal{I}^{(i)} : d\alpha \equiv 0 \pmod{\mathcal{I}^{(i)}}\}.$$

Then, the derived flag of a Pfaff system \mathcal{I} is defined inductively by the exact sequence

$$0 \longrightarrow \mathcal{I}^{(i+1)} \longrightarrow \mathcal{I}^{(i)} \longrightarrow d\mathcal{I}^{(i)} \left(\pmod{\mathcal{I}^{(i)}} \right) \longrightarrow 0.$$

If the codimension of each Pfaff system $\mathcal{I}^{(i)}$ is generically constant then there will be an integer N such that $\mathcal{I}^{(N)} = \mathcal{I}^{(N+1)}$. This integer N is called the *derived length* of \mathcal{I} . The Pfaff system $\mathcal{I}^{(N)}$ is always integrable by definition since

$$d\mathcal{I}^{(N)} \equiv 0 \pmod{\mathcal{I}^{(N)}}.$$

If $\mathcal{I}^{(N)} = 0$ we say that the system \mathcal{I} is *completely nonholonomic*. A contact system on $(\mathbb{C}^3, 0)$ is a completely nonholonomic system. In this work we are interested on completely nonholonomic system on $(\mathbb{C}^4, 0)$ of codimension 2 and derived length equal to 2.

Definition 1.1. A germ of Engel system in $(\mathbb{C}^4, 0)$ is a system $\mathcal{I} = \langle \alpha, \beta \rangle$ of codimension 2 in $(\mathbb{C}^4, 0)$ satisfying the following conditions:

- (i) $\alpha \wedge \beta \wedge d\alpha \neq 0$
- (ii) $\alpha \wedge \beta \wedge d\beta \equiv 0$
- (iii) $\beta \wedge d\beta \neq 0$,

We can see that a germ of *Engel system* in $(\mathbb{C}^4, 0)$ is a system of codimension 2 such that, for $0 \leq i \leq 2$, the elements of its derived flag satisfy $\text{cod}(\mathcal{I}^{(i)}) = 2 - i$. In fact, $\mathcal{I}^{(0)} = \langle \alpha, \beta \rangle$, $\mathcal{I}^{(1)} = \langle \beta \rangle$ and $\mathcal{I}^{(2)} = 0$. Thus, an Engel system has derived length equal to 2.

In the real non-singular case, these Pfaff systems were introduced by E. von Weber in 1898 and studied by several authors [4][6][7][12]. F. Engel [6] shows that a non singular Engel system is locally isomorphic, at a generic point, to the canonical system

$$\mathcal{I}_0 = \langle dz_4 - z_3 dz_1, dz_3 - z_2 dz_1 \rangle.$$

M. Zhitomirskii in [15] obtained normal forms for nonsingular Engel along non generic points.

The canonical system appears naturally as a system called canonical contact system on the space $J^2(\mathbb{C}, \mathbb{C})$ of 2-jets of holomorphic maps of \mathbb{C} , see [12]. Non-singular Global holomorphic Engel systems have been studied by L. Solá Conde and F. Presa in [13].

We prove the following result for germs of holomorphic Engels system on $(\mathbb{C}^4, 0)$

Theorem 1.1. Let \mathcal{I} be a germ of holomorphic Engel system on $(\mathbb{C}^4, 0)$ with $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$. Then there exist $f_1, \dots, f_4 \in \mathcal{O}_0^4$ such that

$$\mathcal{I} = \langle df_4 - f_3 df_1, df_3 - f_2 df_1 \rangle.$$

More precisely, there exists a germ of holomorphic map $f := (f_1, f_2, f_3, f_4) : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^4, 0)$ which is a biholomorphism outside $\text{Sing}(\mathcal{I}) \cup \text{Sing}(d\mathcal{I}^{(1)})$ such that $f^*\mathcal{I}_0 = \mathcal{I}$.

We note that the only place the dimension assumption is used in the proof is to guarantee that $\beta \wedge (d\beta)^2 \equiv 0$. Therefore, we obtain the following result which will be useful in application to Engel systems on four dimensional projective spaces.

Corollary 1.1. *Let $\mathcal{I} = \langle \alpha, \beta \rangle$ be a Pfaff system of codimension 2 on $(\mathbb{C}^n, 0)$ satisfying the following conditions:*

- (i) $\alpha \wedge \beta \wedge d\alpha \neq 0$
- (ii) $\alpha \wedge \beta \wedge d\beta \equiv 0$
- (iii) $\beta \wedge d\beta \neq 0$ and $\beta \wedge (d\beta)^2 \equiv 0$

If $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$, then there exist $f_1, \dots, f_4 \in \mathcal{O}_0^n$ such that

$$\mathcal{I} = \langle df_4 - f_3df_1, df_3 - f_2df_1 \rangle.$$

An interesting consequence of Theorem 1.1 is the existence of an integral analytic curve passing through the singular points of the system.

Theorem 1.2. *Let \mathcal{I} be a germ of holomorphic Engel system on $(\mathbb{C}^4, 0)$ with $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$, then there exists a germ of an analytic curve passing through $\text{Sing}(\mathcal{I})$ which is a solution of \mathcal{I} .*

Proof. In fact, it follows from Theorem 1.1 that the analytic curve $\{f_1 = f_3 = f_4 = 0\}$ is a solution of $\mathcal{I} = \langle df_4 - f_3df_1, df_3 - f_2df_1 \rangle$. \square

To prove Theorem 1.1 we use the singular version of Pfaff's Theorem due to D. Cerveau [5]. Before, we need to define the class of a 1-form as follows.

Let β be a germ of holomorphic 1-form in $(\mathbb{C}^n, 0)$. We define the *class* of the β to be the integer r for which

$$\beta \wedge (d\beta)^r \neq 0, \quad \beta \wedge (d\beta)^{r+1} \equiv 0.$$

Theorem 1.3. [5, Pfaff-Cerveau] *Let β be a germ of holomorphic 1-form on $(\mathbb{C}^n, 0)$ of class r and $\text{cod}(\text{Sing}(d\beta)) \geq 3$. Then there exist $f_1, \dots, f_{r+1}, g_1, \dots, g_r \in \mathcal{O}_0^n$ such that*

$$\beta = \sum_{i=1}^r f_i dg_i + df_{r+1}.$$

2. APPLICATION TO ENGEL SYSTEMS ON PROJECTIVE SPACES

A Pfaff system \mathcal{I} of codimension k on a complex projective space \mathbb{P}^n is a locally decomposable section

$$\omega_{\mathcal{I}} \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^k \otimes \mathcal{L}).$$

. This means that for all $p \in \mathbb{P}^n$ there exists a neighborhood U of p and 1-forms $\omega_1, \dots, \omega_k \in \mathbb{P}^n, \Omega_U^1$, such that $\omega_{\mathcal{I}}|_U = \omega_1 \wedge \dots \wedge \omega_k$.

If $i : \mathbb{P}^k \rightarrow \mathbb{P}^n$ is a generic linear immersion then $i^*\omega_{\mathcal{I}} \in H^0(\mathbb{P}^k, \Omega_{\mathbb{P}^k}^k \otimes \mathcal{L})$ is a section of a line bundle, and its zero divisor reflects the tangencies between \mathcal{I}

and $i(\mathbb{P}^k)$. The *degree* of \mathcal{I} is, by definition, the degree of such a tangency divisor. Set $d := \deg(\mathcal{I})$. Since $\Omega_{\mathbb{P}^k}^k \otimes \mathcal{L} = \mathcal{O}_{\mathbb{P}^k}(\deg(\mathcal{L}) - k - 1)$, one concludes that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d + k + 1)$.

We say that \mathcal{I} is *globally decomposable* if $\omega_{\mathcal{I}} = \omega_1 \wedge \cdots \wedge \omega_k$.

Besides, the Euler sequence implies that a section ω of $\Omega_{\mathbb{P}^n}^k(d + k + 1)$ can be thought of as a polynomial k -form on \mathbb{C}^{n+1} with homogeneous coefficients of degree $d + 1$, which we will still denote by ω , satisfying

$$(1) \quad i_R \omega = 0$$

where

$$R = x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$$

is the radial vector field. Thus the study of distributions of degree d on \mathbb{P}^n reduces to the study of locally decomposable homogeneous k -forms of degree $d + 1$ on \mathbb{C}^{n+1} satisfying the relation (1).

We will use the following Jouanolou's Lemma.

Lemma 2.1. [8, Lemme 1.2, pp. 3] *If η is a homogeneous q -form of degree s , then*

$$i_R d\eta + d(i_R \eta) = (q + s)\eta$$

where R is the radial vector field and i_R denotes the interior product or contraction with R .

All codimension one integrable systems on \mathbb{P}^n have in its singular set an irreducible component of codimension two.

Theorem 2.1. [8] [9] *Let \mathcal{I} be a codimension one integrable system on \mathbb{P}^n , $n \geq 3$, such that $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$. Then $\text{Sing}(\mathcal{I})$ has an irreducible component of codimension two.*

Let \mathcal{I} be a codimension one integrable system on a Fano manifold such that $\text{Sing}(\mathcal{I}) \neq \emptyset$. In [10, Corollary 4.7] F. Lora, J. V. Pereira and F. Touzet show that if \mathcal{I} has a canonical class numerically trivial then its singular set has a component of codimension two.

A similar situation appears in the study of singularities of Poisson structures on Fano manifolds motivated by Bondal's conjecture [2] [1, Conjecture 4]. A. Polishchuk in [14] showed that the rank of a nondegenerate Poisson structure on a Fano variety of odd dimension drops along the subset of codimension two.

As an application of Theorem 1.1 we prove that a globally decomposable Engel system on four dimensional projective space has a singular set with atypical codimension. In fact, the expected codimension of the singular set of a codimension 2 should be 3. But, the following Theorem shows that the singular set of these systems has codimension ≤ 2 .

Theorem 2.2. *Let \mathcal{I} be a globally decomposable holomorphic Engel system on \mathbb{P}^4 . Then, either $\text{Sing}(d\mathcal{I}^{(1)})$ has a component of codimension two, or $\text{Sing}(\mathcal{I})$ has a component of codimension one. Moreover, if $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$, then $\text{Sing}(\mathcal{I})$ has a component of codimension two.*

Proof. Firstly we observe that on \mathbb{P}^4 all holomorphic 1-forms $\beta \in H^0(\mathbb{P}^4, \Omega_{\mathbb{P}^4}^1(s))$ satisfy $\beta \wedge (d\beta)^2 = 0$. In fact, $\beta \wedge (d\beta)^2$ is a 5-form on \mathbb{P}^4 .

Suppose that $\text{Cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{Cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$. Then, it follows from Corollary 1.1 that there exist homogeneous polynomials f_1, f_2, f_3, f_4 on \mathbb{C}^5 such that

$$\alpha = df_4 - f_3 df_1, \beta = df_3 - f_2 df_1.$$

In particular, we have that $\beta \wedge d\beta = -df_4 \wedge df_3 \wedge df_1$ and $\alpha \wedge \beta \wedge d\alpha = df_1 \wedge df_2 \wedge df_3 \wedge df_4$.

Since $i_R \alpha = i_R \beta = 0$, we conclude that $k_4 f_4 - k_1 f_3 f_1 = k_3 f_3 - k_1 f_2 f_1 = 0$, where $k_i = \deg(f_i)$, $i = 1, 3, 4$. These relations imply that

$$\beta \wedge d\beta = \alpha \wedge \beta \wedge d\alpha \equiv 0.$$

This is a contradiction. On the other hand, suppose that $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$. By lemma 2.1 we have

$$i_R(d\beta) = (s+1)\beta$$

since $i_R \beta = 0$. Using this relation we have that

$$\text{Sing}(d\beta) \subset \text{Sing}(i_R(d\beta) \wedge \alpha) = \text{Sing}((s+1)\beta \wedge \alpha) = \text{Sing}(\beta \wedge \alpha).$$

We conclude that $\text{Sing}(\mathcal{I})$ has a component of codimension two. \square

Example 1. Consider the differential system induced by the 1-forms

$$\alpha = z_0^2 dz_4 - z_0 z_3 dz_1 + (z_1 z_3 - z_0 z_4) dz_0$$

and

$$\beta = z_0^2 dz_3 - z_0 z_2 dz_1 + (z_1 z_2 - z_0 z_3) dz_0.$$

A calculation shows that the pair of 1-forms (α, β) satisfy the conditions *i*), *ii*) and *iii*) of definition 1.1 and $i_R \alpha = i_R \beta = 0$. Therefore, the differential system $\mathcal{I} = \langle \alpha, \beta \rangle$ induces a decomposable Engel system on \mathbb{P}^4 .

We have that

$$\begin{aligned} \alpha \wedge \beta &= z_0^4 dz_4 \wedge dz_3 - z_0^3 z_2 dz_4 \wedge dz_1 + z_0^2 (z_1 z_2 - z_0 z_3) dz_4 \wedge dz_0 - \\ &\quad - z_0^3 z_3 dz_1 \wedge dz_3 - z_0 z_3 (z_1 z_2 - z_0 z_3) dz_1 \wedge dz_0 + \\ &\quad + z_0^2 (z_1 z_3 - z_0 z_4) dz_0 \wedge dz_3 - z_0 z_2 (z_1 z_3 - z_0 z_4) dz_0 \wedge dz_1. \end{aligned}$$

Therefore $\text{Sing}(\alpha \wedge \beta) = \{z_0 = 0\}$ has codimension one. Moreover,

$$\text{Sing}(d\beta) = \{z_0 = z_1 = z_2 = 0\}$$

has codimension 3.

Example 2. Consider the differential system induced by the 1-forms

$$\alpha = z_0^3 dz_1 + z_3^2 z_0 dz_4 - (z_0^2 z_1 - z_3^2 z_4) dz_0$$

and

$$\beta = z_0^3 dz_2 + z_3 z_4 z_0 dz_4 - (z_0^2 z_2 + z_3 z_4^2) dz_0.$$

A calculation show that the pair of 1-forms (α, β) satisfies the conditions *i*), *ii*) and *iii*) of definition 1.1 and $i_R \alpha = i_R \beta = 0$. Therefore, the differential system $\mathcal{I} = \langle \alpha, \beta \rangle$ induces a decomposable Engel system on \mathbb{P}^4 . We can see that

$$\text{Sing}(\alpha \wedge \beta) = \{z_0 = 0\} \text{ and } \text{Sing}(d\beta) = \{z_0 = z_3 = z_4 = 0\}.$$

3. PROOF OF THE THEOREM 1.1

Proof. Since $d\beta \wedge \beta \neq 0$ and $(d\beta)^2 \wedge \beta \equiv 0$, we have that β has class 1. By Theorem 1.3, there exist $f_1, f_3, f_4 \in \mathcal{O}_0^4$ such that

$$\beta = df_4 - f_3 df_1.$$

In particular, $d\beta = df_1 \wedge df_3$. Now, since $d\beta \wedge \alpha \wedge \beta = 0$ we get

$$0 = d\beta \wedge \alpha \wedge \beta = df_1 \wedge df_3 \wedge \alpha \wedge \beta.$$

Since $\text{CodSing}(d\beta)$. This implies that there exist germs of holomorphic functions \tilde{a}, \tilde{b} and $\tilde{\lambda}$ on $U = (\mathbb{C}^4, 0) - \text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$ such that

$$\alpha|_U = \tilde{a} df_1 + \tilde{b} df_3 + \tilde{\lambda} \beta.$$

Since the codimension of $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$ is bigger than 2, by Hartogs's extension Theorem we have the identity

$$\alpha - \lambda \beta = a df_1 + b df_3, \text{ where } a, b, \lambda \in \mathcal{O}_0^4$$

on $(\mathbb{C}^4, 0)$. Now if either $a = 0$ or $b = 0$, then $\alpha \wedge \beta \wedge d\alpha \equiv 0$, a contradiction to Engel's conditions. Thus $a \neq 0$ and $b \neq 0$, and therefore

$$\frac{1}{b} \alpha - \frac{\lambda}{b} \beta = \frac{a}{b} df_1 + df_3$$

and if we set $f_2 = -\frac{a}{b}$ then

$$\frac{1}{b} \alpha - \frac{\lambda}{b} \beta = df_3 - f_2 df_1.$$

Thus,

$$\mathcal{I} = \langle \alpha, \beta \rangle = \left\langle \alpha, \frac{1}{b} \alpha - \frac{\lambda}{b} \beta \right\rangle = \langle df_4 - f_3 df_1, df_3 - f_2 df_1 \rangle.$$

Now, we will prove that the map $f : (f_1, f_2, f_3, f_4) : (\mathbb{C}^4, 0) \rightarrow$ is a biholomorphism outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$. That is, we prove that

$$df_1 \wedge df_2 \wedge df_4 \wedge df_3$$

never vanishing outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$. Differentiating the identity

$$\frac{1}{b} \alpha = \frac{a}{b} df_1 + df_3 + \frac{\lambda}{b} \beta.$$

we get

$$d \left(\frac{1}{b} \right) \wedge \alpha + \frac{1}{b} d\alpha = d \left(\frac{a}{b} \right) \wedge df_1 + d \left(\frac{\lambda}{b} \right) \wedge \beta + \frac{\lambda}{b} d\beta.$$

Multiplying this identity by $\beta \wedge \alpha$ we obtain

$$\left[d \left(\frac{a}{b} \right) \wedge df_1 + d \left(\frac{\lambda}{b} \right) \wedge \beta + \frac{\lambda}{b} d\beta \right] \wedge \beta \wedge \alpha = d \left(\frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha$$

since $d\beta \wedge \beta \wedge \alpha \equiv 0$. Thus

$$d\left(\frac{a}{b}\right) \wedge df_1 \wedge \beta \wedge \alpha = \left[d\left(\frac{1}{b}\right) \wedge \alpha + \frac{1}{b}d\alpha \right] \wedge \beta \wedge \alpha.$$

Then

$$d\left(\frac{a}{b}\right) \wedge df_1 \wedge \beta \wedge \alpha = \frac{1}{b}d\alpha \wedge \beta \wedge \alpha \neq 0.$$

Using that $\alpha = adf_1 + bdf_3 + \lambda\beta$ and $\beta = df_4 - f_3df_1$ and substituting in $d\left(\frac{a}{b}\right) \wedge df_1 \wedge \beta \wedge \alpha$ we conclude that

$$0 \neq d\left(\frac{a}{b}\right) \wedge df_1 \wedge \beta \wedge \alpha = bd\left(\frac{a}{b}\right) \wedge df_1 \wedge df_4 \wedge df_3 = bdf_2 \wedge df_1 \wedge df_4 \wedge df_3$$

never vanishing outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$. \square

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